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HAUSDORFF DIMENSIONS OF SOFIC AFFINE-INVARIANT SETS

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ABSTRACT

We determine the Hausdorff and Minkowski dimensions of compact subsets of the 2-torus which are invariant under a linear endomorphism with integer eigenvalues and correspond to shifts of finite type or sofic shifts via some Markov partition. This extends a result of McMullen (1984) and Bedford (1984), who considered full-shifts.

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1. Introduction and main results

McMullen [25] and Bedford [2] independently studied plane sets constructed as follows. Let $1 < m \leq n$ be integers. By drawing $n-1$ vertical lines and $m-1$ horizontal lines, partition the unit square into nm congruent rectangles. Let S be a subcollection of these rectangles; erase all other rectangles and partition the remaining ones into nm congruent subrectangles, again keeping only those which correspond to the pattern S. Repeating this procedure ad infinitum, a compact set is obtained. Viewed as subsets of the 2-torus, such sets are invariant under the toral endomorphism $\begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}$, and are the simplest invariant sets: using the natural Markov partition obtained by expanding in base n along the x-axis and in base m along the y-axis, such a set corresponds to a full-shift on $|S|$ symbols. Nevertheless, when $n > m$ the different expansion coefficients in the horizontal and vertical directions make the quantitative analysis of these sets quite delicate; McMullen and Bedford succeeded in calculating their Hausdorff dimension and showing it is usually strictly smaller than the Minkowski (i.e. Box) dimension. For instance, the set determined by the pattern in Figure 1 has Hausdorff dimension $\log_2(1 + 2^{\log_3 2})$; for a general pattern S, the dimension is $\log_m(\sum_{i=0}^{m-1} z(i)^{\log_n m})$ where $z(i)$ is the number of rectangles in row i of S.

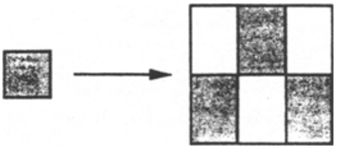

Fig. 1. Typical McMullen's example

Apart from their intrinsic interest, the motivation for studying these sets is twofold: First, their analysis leads to questions concerning the asymptotics of matrix products which arise in other contexts, e.g. in Kesten [21]; see section 3. More importantly, these sets may serve as prototypes for invariant sets under nonconformal smooth maps.

Now we concentrate on subsets of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ which are invariant under the endomorphism

$$T = \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}$$

and correspond to shifts of finite type. Denote by

$$D = \{0, \ldots, n-1\} \times \{0, 1, \ldots, m-1\}$$

the set of digits. To every sequence $\{d_k\}_{k=1}^{\infty}$ in $D^{\mathbb{N}}$ there corresponds a point in $\mathbb{T}^2$ via what may be called "base T representation":

$$R_T(\{d_k\}) = \sum_{k=1}^{\infty} \begin{pmatrix} n^{-k} & 0 \\ 0 & m^{-k} \end{pmatrix} d_k. \tag{1}$$

Any $0-1$ matrix A with rows and columns indexed by D defines a shift of finite type and we let $K_T(A)$ be its image under R_T, i.e.

$$K_T(A) = \{R_T\{d_k\} \mid A(d_k, d_{k+1}) = 1 \text{ for } k \geq 1\}. \tag{2}$$

THEOREM 1.1: *Given the endomorphism T, the $0-1$ matrix A and the compact T-invariant set $K_T(A)$ defined above, construct m matrices $A_0, A_1, \ldots, A_{m-1}$ with rows and columns indexed by D, as follows: $A_j(d,d') = A(d,d')$ if the second coordinate of $d' \in D$ is j, and $A_j(d,d') = 0$ otherwise. Then*

$$\dim K_T(A) = \lim_{k\to\infty} \frac{1}{k} \log_m \sum_{0 \leq i_1,\ldots,i_k \leq m-1} \|A_{i_k} \cdot A_{i_{k-1}} \cdot \ldots \cdot A_{i_1}\|^{\alpha} \tag{3}$$

where dim *denotes Hausdorff dimension and $\alpha = \log m / \log n \leq 1$ (the choice of norm is clearly immaterial).*

The theorem is proved in a more general form (pertaining to sofic systems) in section 3 where methods for calculating the limit in (3) are discussed. Explicit calculations are in section 4.

Example: Let K be the set determined by starting with a square a and iterating the substitution rules in Figure 2. Let $\alpha = \log 2/\log 3$. Then the Hausdorff dimension of K is $\log_2 r \simeq 0.883127\ldots$ where r is the unique positive solution of the equation

$$r = \frac{1}{r} + \frac{2^{\alpha}}{r^2} + \frac{3^{\alpha}}{r^3} + \cdots \tag{4}$$

Equivalently, r is the spectral radius of the infinite matrix

$$\begin{pmatrix} 0 & 1 & 2^{\alpha} & 3^{\alpha} & 4^{\alpha} & \ldots \\ 1 & 0 & 0 & \ldots & & \\ 0 & 1 & 0 & \ldots & & \\ 0 & 0 & 1 & 0 & \ldots & \\ \vdots & & & \ddots & & \end{pmatrix}.$$

The Minkowski dimension of K is approximately 0.8875138...; observe that this is quite close to the Hausdorff dimension. The details are in Example 4.2.

Fig. 2. A sofic invariant set.

Recall that the **Minkowski dimension** (sometimes called **box dimension**) of a totally bounded set K in a metric space is

$$\dim_M(K) = \lim_{\varepsilon \downarrow 0} \frac{\log \eta(K,\varepsilon)}{-\log \varepsilon}$$

when the limit exists, where $\eta(K,\epsilon)$ is the smallest number of ε-balls which cover K.

The Minkowski dimension is usually easier to calculate than the Hausdorff dimension. The next proposition is contained in theorem 3 of Deliu et al. [12].

PROPOSITION 1.2: *Assume the adjacency matrix A is* **primitive**, *i.e. some power of A has positive entries. Then the set $K_T(A)$ considered in the previous theorem has Minkowski dimension*

$$\begin{aligned}\dim_M(K_T(A)) &= \frac{1}{\log n} h_{top}(T, K_T(A)) \\ &+ \left(\frac{1}{\log m} - \frac{1}{\log n}\right) h_{top}(y \to my \bmod 1, \pi_Y(K_T(A))),\end{aligned}$$

where π_Y is projection to the y-axis and $h_{top}(T,K)$ denotes the topological entropy of the map T on the compact set K.

Remark: The topological entropy $h_{top}(T, K_T(A))$ equals the logarithm of the spectral radius of A (Parry [26]). Since the map $y \to my \bmod 1$ on $\pi_Y K_T(A)$ is a finite-to-one factor of a sofic system (cf. Weiss [30] and §3), its topological entropy can also be written as log of the spectral radius of a suitable integer matrix.

When do the Minkowski and Hausdorff dimensions coincide?

For the sets considered in McMullen [25], this happens only if in the pattern S defining the construction, all nonempty rows have the same number of rectangles. This is explained and extended in the next theorem.

THEOREM 1.3: *Assume that the matrix A is primitive. For the set $K_T(A)$ considered in Theorem* 1.1, $\dim[K_T(A)] = \dim_M[K_T(A)]$ *iff the unique invariant measure of maximal entropy on $K_T(A)$ projects via π_Y to the unique measure of maximal entropy on $\pi_Y(K_T(A))$.*

Note that the measures of maximal entropy for shifts of finite type (and sofic systems) are given by an explicit formula (due to Parry [26]; see Walters [29]) so the condition of the theorem can be checked.

The rest of the paper is organized as follows. In the next section, we survey some of the previous work on self-affine sets. In section 3, Theorem 1.1 and Proposition 1.2 are proved.

Section 4 is devoted to examples pertaining to Theorems 1.1, 1.2, and 1.3. Considerable simplification in the formula (3) for the dimension of $K_T(A)$ occurs if the matrices A_j appearing there share the same Perron eigenvector or if one of them has rank 1. The first case includes the self affine graphs studied by Kono [22], Bedford [4, 5] and Urbanski [27, 28] (cf. Example 4.1); the construction appearing in Figure 2 illustrates the second case (cf. Example 4.2).

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2. Background

An attractive introduction to Hausdorff dimension is Falconer [14]. In particular, following Hutchinson, it is shown there that for "self-similar" sets, the Hausdorff and Minkowski dimensions coincide. The same coincidence was proved by Furstenberg [16] for compact subsets of the circle which are invariant under an endomorphism $x \to nx \bmod 1$. Furstenberg's result extends to compact subsets of higher dimensional tori which are invariant under **conformal** toral endomorphisms; see Dekking [11], Mauldin and Williams [24] for further extensions and refinements in the self-similar setting. When the conformality assumption is removed, McMullen and Bedford showed that the Hausdorff dimension of an invariant set may be strictly smaller than the Minkowski dimension. Indeed, this is the typical situation for the self-affine carpets

$$K(T,D) = \left\{ \sum_{k=1}^{\infty} \begin{pmatrix} n^{-k} & 0 \\ 0 & m^{-k} \end{pmatrix} d_k \mid d_k \in D \right\}$$

(where $D \subset \{0,1,\ldots,n-1\} \times \{0,1,\ldots,m-1\}$) which are invariant under the endomorphism

$$T \simeq \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}.$$

The set $K(T,D)$ can also be viewed as the attractor for the family of $|D|$ affine contractions

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} n^{-1} & 0 \\ 0 & m^{-1} \end{pmatrix} \left[\begin{pmatrix} x \\ y \end{pmatrix} + d \right] : d \in D \right\},$$

i.e. $K(T,D)$ is the unique compact set which is the union of its images under these contractions. This point of view is adopted in Falconer [13, 15] and in Gatzouras and Lalley [17].

Falconer considers a family of affine maps $\{L_i(x) + a_i\}_{i=1}^k$ where the L_i are linear contractions of $\mathbf{R}^d$ with $\|L_i\| < \frac{1}{3}$ and $a_i \in \mathbf{R}^d$. He shows that for Lebesgue almost all translations $\{a_i\}_{i=1}^k$ the attractor of this family has equal Hausdorff and Minkowski dimensions and gives an asymptotic formula for this dimension. The examples considered by McMullen, Bedford, Gatzouras and Lalley as well as those in the present paper (cf. Example 4.4) typically fall in the exceptional set for Falconer's theorem. It would be nice to have checkable conditions on the translations a_i for the validity of his formula. Gatzouras and Lalley extend McMullen's theorem to attractors of families of affine contractions which map the unit square to rectangles contained in it with height greater than width, such that these rectangles are lined up in rows — for any two of them, the projections to the y-axis are disjoint or identical. They show that the attractor supports a "Bernoulli" measure (obtained by assigning suitable probabilities to the affine maps) which has full Hausdorff dimension. In Example 4.4, we show this no longer holds if the rectangles referred to above are not "lined up".

The Hausdorff dimension estimates in this paper are based on the following basic lemma (cf. Billingsley [7], Young [31], or Falconer [14, Prop. 4.9]).

BILLINGSLEY'S LEMMA: *Let μ be a positive finite Borel measure on the r-torus $\mathbf{T}^r$. Assume $K \subset \mathbf{T}^r$ is a Borel set satisfying $\mu(K) > 0$ and*

$$K \subset \left\{ x \in \mathbf{T}^r \mid \liminf_{\varepsilon \downarrow 0} \frac{\log \mu[B_\varepsilon(x)]}{\log \varepsilon} = \gamma \right\}.$$

Then $\dim(K) = \gamma$.

Here the balls $B_\varepsilon(x)$ may be replaced by cubes, etc.

Define the dimension of a measure, $\dim(\mu)$, as the minimal dimension of Borel sets of full measure. If a measure μ satisfies

$$\liminf_{\varepsilon\downarrow 0} \frac{\log \mu[B_\varepsilon(x)]}{\log \varepsilon} = \gamma \quad \text{for } \mu\text{-a.e. } x,$$

then $\dim(\mu) = \gamma$ and furthermore any set of dimension $< \gamma$ has μ-measure zero.

3. Dimensions of sofic sets

Our first aim is to prove a slightly generalized form of Theorem 1.1. Subshifts which are factors of shifts of finite type (SFT's) were called **sofic systems** by Weiss [30] (a **factor** map is a continuous, shift-commuting map). Boyle, Kitchens and Marcus [10] showed that the factor map can be taken to be "right resolving" which yields the following.

PROPOSITION 3.1: *Let $G = \langle V, E\rangle$ be a finite directed graph in which loops and multiple edges are allowed. Suppose the edges of G are colored in ℓ colors in a right-resolving fashion: no two edges emanating from the same vertex have the same color. Then the color sequences which arise from infinite paths in G form a sofic system on ℓ symbols. Conversely, any sofic system may be generated in this way.*

The right-resolving property of the coloring implies that every color sequence arises from at most $|V|$ paths in G. *This is crucial below.* The edges in G are conveniently represented by the **adjacency matrix** A where for any two vertices v, w in G, $A(v, w)$ is the number of edges in E from v to w.

For the rest of this section, let T be the toral endomorphism $\begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}$.

Definition: The image of a sofic system on the nm symbols $\{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, m-1\}$ by the representation map R_T defined in (1) is called a **T-invariant Sofic set.**

The notion of a sofic set is more flexible than it seems. A sofic set corresponds to a sofic system in any Markov partition (cf. Ashley, Kitchens and Stafford [1]). Also, for any sofic system with symbols in $\mathbb{Z}^2$, its image under R_T is a sofic set; this is proved exactly like Theorem 5.5 in Kenyon and Peres [20]. We now state the extension of Theorem 1.1.

THEOREM 3.2: *Let G be a directed graph as in Proposition 3.1 with edges labeled by $\{0,\ldots,n-1\}\times\{0,\ldots,m-1\}=D$ and adjacency matrix A. Let $\Omega\subset D^{\mathbb{N}}$ be the resulting sofic system. Then the Hausdorff dimension of $R_T(\Omega)$ is given by the right-hand side of (3), where for $0\le j<m$ and vertices v,v' in G, $A_j(v,v')$ denotes the number of edges from v to v' in G such that the second coordinate of their label is j.*

Remarks: 0. To get a vivid picture of the matrices A_j, see the examples in section 4.

1. Observe that $\sum_{j=0}^{m-1} A_j = A$. To obtain Theorem 1.1 from Theorem 3.2, simply restrict attention to graphs with vertex set D and no multiple edges, with every edge labeled by the vertex it leads to.

2. If we use a norm satisfying $\|B_1B_2\|\le\|B_1\|\cdot\|B_2\|$ on matrices, then the expression $\log\sum_{0\le j_1,\ldots,j_N<m}\|A_{j_1}\cdot\ldots\cdot A_{j_N}\|^\alpha$ is subadditive in N so the limit in (3) exists. Consequently, it exists for any norm.

Proof of Theorem 3.2: The upper bound on the dimension of $R_T(\Omega)$ is easier to prove. Let N be a large integer and consider the set $\Omega^{(N)}$ consisting of sequences $\{\omega_\nu\}_{\nu=1}^\infty$ such that each block $(\omega_{kN+1},\omega_{kN+2},\ldots,\omega_{(k+1)N})$ for $k\ge 0$ can be extended to an infinite sequence in Ω.

Then $R_T(\Omega^{(N)})$ is a McMullen carpet with respect to the endomorphism T^N. The number of rectangles in row $\sum_{\nu=1}^N j_\nu m^{N-\nu}$ of the initial pattern defining this carpet is bounded above by $\|A_{j_1}A_{j_2}\cdot\ldots\cdot A_{j_{N-1}}A_{j_N}\|$ (where we use as norm the sum of the absolute values of the elements) since each such rectangle corresponds to at least one path of length N in G. By McMullen's theorem

$$\begin{aligned}\dim R_T(\Omega) &\le \dim R_T(\Omega^{(N)})\\ &\le \frac{1}{\log(m^N)}\log\sum_{0\le j_1,\ldots,j_N<m}\|A_{j_1}\ldots A_{j_N}\|^\alpha\end{aligned}$$

with

$$\alpha=\frac{\log(m^N)}{\log(n^N)}=\frac{\log m}{\log n}.$$

The lower bound for $\dim[R_T(\Omega)]$ is trickier. First, for the second inequality above, given $j_1,j_2,\ldots,j_N$ in $\{0,1,\ldots,m-1\}$ the number of blocks $(\omega_1,\omega_2,\ldots,\omega_N)$ which are legal in Ω *and* project to $(j_1,\ldots,j_N)$ in the second coordinate is at least $\frac{1}{|V|}\|A_{j_1}\cdots A_{j_N}\|$ where V is the vertex set of G. Hence $\lim_{N\to\infty}\dim R_T(\Omega^{(N)})$ indeed equals the limit in (3).

The remainder of the proof relies on the following general lemma on matrices.

LEMMA 3.3: *Given m nonnegative matrices $A_0, A_1, \ldots, A_{m-1}$ with rows and columns indexed by the finite set V and given $0 < \alpha \le 1$, define for $v \in V$*

$$\varphi_N(v) = \sum_{0 \le j_1,\ldots,j_N < m} [(A_{j_1} \cdot A_{j_2} \cdot \ldots \cdot A_{j_N})(v,v)]^\alpha$$

and

$$\Phi_N = \sum_{0 \le j_1,\ldots j_N < m} \|A_{j_1} \cdot \ldots \cdot A_{j_N}\|^\alpha$$

where $\|B\| = \sum_{v,w \in V} |B(v,w)|$.

Then there exists a vertex $v \in V$ such that

$$\limsup_{N\to\infty} [\varphi_N(v)]^{1/N} = \lim_{N\to\infty} \Phi_N^{1/N}$$

Proof: For any sequence $(v_0, v_1, \ldots, v_N)$ of $N+1$ elements of V, define its **skeleton** to be $\begin{pmatrix} 0 \\ v_0 \end{pmatrix}, \begin{pmatrix} s_1 \\ v_{s_1} \end{pmatrix}, \begin{pmatrix} s_2 \\ v_{s_2} \end{pmatrix}, \ldots, \begin{pmatrix} s_\ell \\ v_{s_\ell} \end{pmatrix}$ where $v_{s_1 - 1} = v_0$ is the last occurrence of v_0 in the sequence, v_{s_2-1} the last occurrence of v_{s_1}, etc.; finally s_ℓ is the smallest s for which $v_s = v_N$. For instance, if $v_0 = v_N$ the skeleton is just $\begin{pmatrix} 0 \\ v_0 \end{pmatrix}$. Write momentarily $B_\ell = A_{j_\ell}$ and expand

$$\|B_1 \cdot B_2 \cdot \ldots \cdot B_N\| = \sum_{v_0, v_1, \ldots, v_N} B_1(v_0, v_1) B_2(v_1, v_2) \cdot \ldots \cdot B_N(v_{N-1}, v_N).$$

Next, decompose this sum according to the skeleton of the sequence $v_0, \ldots, v_N$. Using the inequality $(x+y)^\alpha \le x^\alpha + y^\alpha$ we get

$$\|B_1 \cdot B_2 \cdot \ldots \cdot B_N\|^\alpha \le b^{\alpha|V|} \sum_{skeletons} (B_1 \cdot B_2 \ldots B_{s_1-1})(v_0, v_0)^\alpha$$
$$\cdot (B_{s_1+1} \cdot \ldots \cdot B_{s_2-1})(v_{s_1}, v_{s_1})^\alpha \cdot \ldots \cdot (B_{s_\ell+1} \ldots B_N)(v_{s_\ell}, v_{s_\ell})^\alpha$$

where $b = \max_{j,v,v'} A_j(v,v')$ was used to bound the factors from the skeleton itself. Recall that $B_\ell = A_{j_\ell}$ and sum the last inequality over all $0 \le j_1, j_2, \ldots, j_N < m$ to obtain

$$\Phi_N \le b^{\alpha|V|} \sum_{\begin{pmatrix} 0 \\ v_0 \end{pmatrix}, \ldots, \begin{pmatrix} s_\ell \\ v_{s_\ell} \end{pmatrix}} \varphi_{s_1-1}(v_0) \cdot \varphi_{s_2-s_1-1}(v_{s_1}) \cdot \ldots \cdot \varphi_{N-s_\ell-1}(v_{s_\ell}).$$

Consequently, for $t > 0$ the power series

$$\sum_N \Phi_N t^N \tag{5}$$

is bounded above by the product

$$\prod_{v \in V} [b^\alpha \sum_{N=1}^{\infty} \varphi_{N-1}(v) t^N] \tag{6}$$

so the radius of convergence for (5) is at least the minimum over $v \in V$ of the radii of convergence of the factors in (6).

Applying the usual Cauchy–Hadamard formula for the radius of convergence proves the lemma. ∎

Remarks: 1. The lemma above becomes much easier if one assumes that the matrix $\sum_{j=0}^{m-1} A_j$ is primitive.

2. The lemma holds also for $\alpha > 1$; simply use convexity of $t \to t^\alpha$ instead of subadditivity.

Completion of the proof of Theorem 3.2: We use the notation of Lemma 3.3. Fix some $v \in V$ for which

$$\limsup_{N \to \infty} \varphi_N(v)^{1/N} = \lim_{N \to \infty} \Phi_N^{1/N}$$

and consider the collection of sequences $\Omega(N, v)$ which arise as label sequences from infinite paths in the graph $G = \langle V, E \rangle$ and visit v at all times which are multiples of N. Explicitly, given the labeling $L \colon E \to D$, the set $\Omega(N, v)$ consists of all sequences $(L(e_1), L(e_2), L(e_3), \ldots)$ where $(e_1, e_2, e_3, \ldots)$ is a path in G and the initial vertex of e_{qN+1} is v, for all $q \geq 1$.

The set $R_T(\Omega(N, v))$ is a McMullen Carpet for T^N, contained in $R_T(\Omega)$. The number of generating rectangles of size $n^{-N} \times m^{-N}$ in row $\sum_{\nu=1}^{N} i_\nu m^{N-\nu}$ is at least

$$\frac{1}{|V|} A_{i_1} \cdot A_{i_2} \cdot \ldots \cdot A_{i_N}(v, v)$$

and hence by McMullen's formula

$$\dim[R_T(\Omega)] \geq \dim[R_T(\Omega(N, v))] \geq \frac{1}{\log m^N} \log \left[\frac{1}{|V|} \varphi_N(v) \right].$$

Taking limsup as $N \to \infty$ and using our choice of v yields

$$\dim R_T(\Omega) \geq \lim_{N\to\infty} \frac{1}{N\log m} \log \Phi_N.$$

This completes the proof. ∎

Theorem 3.2 leads to the question of calculating the limit in (3). One case is particularly simple.

PROPOSITION 3.4: *Suppose that the nonnegative square matrices $A_0, \ldots, A_{m-1}$ all share the same strictly positive eigenvector u:*

$$A_j u = \lambda_j u.$$

Then:

$$\lim_{N\to\infty} \left[\sum_{0\leq j_1,\ldots,j_N<m} \|A_{j_1}\ldots A_{j_N}\|^\alpha \right]^{\frac{1}{N}} = \sum_{j=0}^{m-1} \lambda_j^\alpha.$$

Proof: Immediate, upon observing that the ratio of $\|A_{j_1}\ldots A_{j_N}\|$ and

$$\|A_{j_1}\ldots A_{j_N}u\| = \lambda_{j_1}\ldots\lambda_{j_N}\cdot\|u\|$$

is bounded. ∎

See Example 4.1 for some examples where Proposition 3.4 applies. In other cases, we resort to spectral theory to study the limit (3).

Following Kesten [21], introduce an operator $\mathcal{L}$ acting on the continuous functions on the sphere $S^{|V|-1}$:

$$(\mathcal{L}f)(x) = \sum_{j=0}^{m-1} \|A_j x\|^\alpha f\left(\frac{A_j x}{\|A_j x\|}\right). \tag{7}$$

Iteration of $\mathcal{L}$ yields

$$(\mathcal{L}^N f)(x) = \sum_{0\leq j_1,\ldots,j_N<m} \|A_{j_1}\ldots A_{j_N}x\|^\alpha f\left(\frac{A_{j_1}\ldots A_{j_N}x}{\|A_{j_1}\ldots A_{j_N}x\|}\right).$$

The usual Gelfand formula implies that the spectral radius $\rho(\mathcal{L})$ is precisely

$$\lim_{N\to\infty} \Big[\sum_{0\leq j_1,\ldots,j_N<m} \|A_{j_1}\ldots A_{j_N}\|^\alpha\Big]^{1/N}. \tag{8}$$

It follows from the results of Kesten [21] and the refinements in Lepage [23] that in many cases the spectrum of $\mathcal{L}$ except for a simple eigenvalue $\rho(\mathcal{L})$ is contained in a disk of radius $\rho(\mathcal{L}) - \varepsilon$ about the origin and hence the convergence in (8) is exponential. For instance, this holds if the matrices A_j are strictly positive — a rare occurrence in our application. It would be useful to know the most general conditions.

The approach indicated above simplifies if one of the matrices $A_0, \ldots, A_{m-1}$, say A_ℓ, has rank 1. We may then replace the sphere by a countable set of directions, the one determined by A_ℓ and its orbit under the semigroup generated by $\{A_j : j \neq \ell\}$. The operator $\mathcal{L}$ reduces to an infinite matrix whose spectral radius can sometimes be calculated explicitly. This is carried out in detail in Example 4.2. The sums appearing in (8) were used by Kesten [21] to study random difference equations and by Lepage [23] to obtain large deviation results for random matrix products; see Bougerol and Lacroix [8] for information on the spectrum of the operator $\mathcal{L}$.

We now present the simple proof of a formula for Minkowski dimension which extends Proposition 1.2. Calculations of this type may be found in Deliu et al. [12].

PROPOSITION 3.5: *Let G be a graph as in Proposition 3.1 with edges labeled by $D = \{0, \ldots, n-1\} \times \{0, \ldots, m-1\}$ and adjacency matrix A. Let $\Omega \subset D^{\mathbf{N}}$ be the resulting sofic system, and let R_T be the representation map defined in (1). Then*

$$\dim_M(R_T(\Omega)) = \frac{\log \rho(A)}{\log n} + \left(\frac{1}{\log m} - \frac{1}{\log n}\right) h_{top}(y \to my \bmod 1, \pi_Y(R_T(\Omega)))$$

where $\rho(A)$ is the spectral radius of A, provided the matrix A is primitive.

Proof: Recall that Ω is obtained from the directed graph $G = \langle V, E\rangle$ and the right-resolving labelling $L\colon E \to \{0, 1, \ldots, n-1\} \times \{0, \ldots, m-1\}$.

Write $L = (L_x, L_y)$ with $L_x\colon E \to \{0, 1, \ldots, n-1\}$ and $L_y\colon E \to \{0, \ldots, m-1\}$.

To every path $e_1, e_2, \ldots, e_k$ in G attach the sequence $L(e_1), \ldots, L(e_k)$ and let $N(k)$ denote the number of sequences so obtained.

Similarly, let $N_Y(k)$ denote the number of distinct sequences of the form

$$L_y(e_1), L_y(e_2), \ldots, L_y(e_k)$$

and $M(k)$ denote the number of distinct sequences of the form

$$L(e_1), L(e_2), \ldots, L(e_{\lfloor \alpha k \rfloor}), L_y(e_{\lfloor \alpha k \rfloor + 1}), \ldots, L_y(e_k)$$

respectively, obtained as $(e_1, e_2, \dots, e_k)$ ranges over all paths of length k in G (here $\alpha = \log m / \log n$). We then have

$$h_{top}(T, R_T(\Omega)) = \lim_{k\to\infty} \frac{1}{k} \log N(k) = \log \rho(A) \tag{9}$$

and

$$h_{top}(y \to my \bmod 1, \pi_Y(R_T(\Omega)) = \lim_{k\to\infty} \frac{1}{k} \log N_Y(k). \tag{10}$$

Also

$$\dim_M(R_T(\Omega)) = \lim_{k\to\infty} \frac{1}{k \log m} \log M(k), \tag{11}$$

provided the last limit exists.

The formulas (9) and (10) follow from Bowen's definition of topological entropy (cf. Bowen [9]) and the inequality $\frac{1}{|V|}\|A^k\| \le N(k) \le \|A\|^k$; (11) is obtained from counting "approximate squares" of size $n^{-\lfloor \alpha k \rfloor} \times m^{-k}$ which intersect $R_T(\Omega)$, as explained in McMullen [25].

Clearly,

$$M(k) \le N(\lfloor \alpha\ k \rfloor) N_Y(k - \lfloor \alpha\ k \rfloor).$$

On the other hand, A^r is a positive integer matrix for some $r \ge 1$ which implies that

$$N(\lfloor \alpha\ k \rfloor - r) N_Y(k - \lfloor \alpha\ k \rfloor) \le M(k).$$

Combining the last two formulae gives

$$\begin{aligned} \frac{1}{k} \log N(\lfloor \alpha k \rfloor - r) + \frac{1}{k} \log N_Y(k - \lfloor \alpha k \rfloor) &\le \frac{1}{k} \log M(k) \\ &\le \frac{1}{k} \log N(\lfloor \alpha k \rfloor) + \frac{1}{k} \log N_Y(k - \lfloor \alpha k \rfloor). \end{aligned}$$

Finally, passing to the limits as $k \to \infty$ here and using (9)–(11) completes the proof. ∎

Remark: The assumption that A is primitive in Propositions 1.2 and 3.5 cannot be dropped.

Let $T = \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}$ and consider the McMullen Carpets $K(T, D_1)$ and $K(T, D_2)$ obtained from

$$D_1 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\} \quad \text{and} \quad D_2 = \left\{ \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}.$$

It is easy to write the union $K(T, D_1) \cup K(T, D_2)$ as $K_T(A)$ for an appropriate $0-1$ matrix A. Clearly, $h_{top}(T, K_T(A)) = \log 3$ and

$$h_{top}(y \to 4y, \pi_Y(K_T(A)) = \frac{1}{2}.$$

Since

$$\dim_M(K_T(A)) = \frac{\log 3}{\log 5},$$

the formula in Proposition 1.2 does not hold. An example constructed in discussions with Shahar Mozes shows the formula need not hold for a T-invariant compact set which is not sofic, even if T is topologically transitive on it.

4. Examples

Example 4.1: Shared Perron eigenvectors. The situation described in Proposition 3.4 occurs in the self-affine graphs studied by Bedford, Kono and Urbanski. In fact consider any substitution rule on 2 symbols which yields the graph of a continuous function, as in Figure 3.

This figure represents the labeled graph of Figure 4.

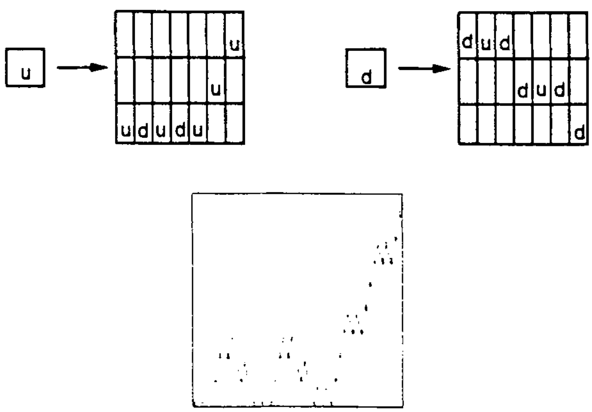

Fig. 3. A continuous, self-affine function.

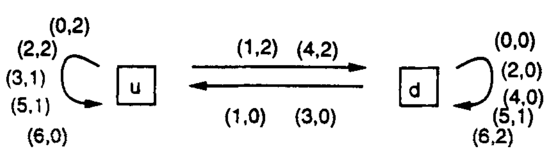

Fig. 4. Sofic system associated with Fig. 3.

Observe that any right resolving labelling as in Proposition 3.1 can be depicted in this manner.

The matrices $A_0, A_1, \dots, A_{m-1}$ constructed in Theorem 1.1 for these examples all have $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as a left eigenvector (e.g. in the example above $A_0 = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$).

Thus Proposition 3.4 yields the dimension of these graphs; McMullen's formula applies verbatim. This was already computed in some special symmetric cases by Bedford [4, 5] and Urbanski [27]; Bedford's method extends to these more general cases and, further, to sofic sets where there exists an invariant measure of maximal dimension which makes the digits independent.

Example 4.2: This example appeared in Figure 2. We read off from there that $A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Let $\alpha = \log_3 2$. Since $A_1 A_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, Theorem 1.1 implies that

$$(12)\ \dim[K_T(A)] = \lim_{N\to\infty} \frac{1}{N} \log_2 \sum_{0\le j_1,\dots,j_N<m} \left[(1,1)A_{j_1}\cdot\dots\cdot A_{j_N}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]^\alpha.$$

To analyze this limit, partition the summands by direction. More precisely, observe that for any N-tuple $(j_1,\dots,j_N)$ the vector $(1,1)A_{j_1}\cdot\dots\cdot A_{j_N}$ is an integer multiple of $(q,1)$ for some nonnegative integer q (we then say that the N-tuple "belongs to q"). Denote by $\Psi_N(q)$ the sum

$$\sum_{0\le j_1,\dots,j_N<m} \left[(1,1)A_{j_1}\cdot\dots\cdot A_{j_N}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]^\alpha$$

extended over all N-tuples which belong to q. By convention, $\Psi_0(0) = 1$ and $\Psi_0(q) = 0$ for $q > 0$. Noting that $(q,1)A_0 = (0,q)$ and $(q,1)A_1 = (q+1,1)$ we get for each $N \ge 0$:

$$\begin{aligned} \Psi_{N+1}(q') &= \Psi_N(q'-1) \quad \text{if } q' \neq 0, \\ \Psi_{N+1}(0) &= \sum_{q=0}^{\infty} \Psi_N(q) q^\alpha. \end{aligned}$$

In vector notation,

$$\Psi_{N+1} = M\Psi_N$$

where M is the infinite matrix, depicted in the introduction, which has ones on the subdiagonal, $(0, 1, 2^\alpha, 3^\alpha, \ldots)$ as the first row and zeros elsewhere.

Define $r > 1$ by the relation

$$r = \sum_{k=1}^{\infty} \frac{k^\alpha}{r^k}.$$

It is immediate that the vector $u = (1, \frac{1}{r}, \frac{1}{r^2}, \ldots)$ satisfies $Mu = ru$. This implies that

$$\|\Psi_N\| = \|M^n \Psi_0\| \le \|M^n u\| = r^n \|u\| \tag{13}$$

(where as usual we use the l^1-norm).

Conversely, apply the Perron–Frobenius theorem to the upper left $\ell \times \ell$ submatrix M_ℓ of M to get

$$\liminf_{N\to\infty} \|\Psi_N\|^{1/N} \ge \liminf_{N\to\infty} \|M_\ell^N \Psi_0\|^{1/N} = r_\ell \tag{14}$$

where r_ℓ, the spectral radius of M_ℓ, satisfies

$$r_\ell = \sum_{k=0}^{\ell-1} \frac{k^\alpha}{r_\ell^k}.$$

Since clearly $\lim_{\ell\to\infty} r_\ell = r$, combining (13) and (14) we obtain

$$\lim_{N\to\infty} \|\Psi_N\|^{1/N} = r.$$

Recalling the definition of Ψ_N, (12) yields $\dim[K_T(A)] = \log_2(r)$ as asserted in the introduction.

To obtain the Minkowski dimension we must find the topological entropy of the projection $\pi_Y(K_T(A))$. This corresponds to a sofic system given by the labelled graph of Figure 5.

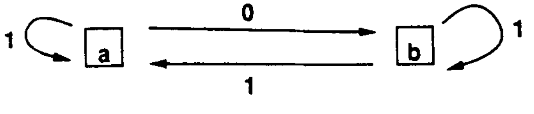

Fig. 5.

This labelling is not right-resolving; however, the same subshift arises if we erase the loop attached to the vertex b (see Boyle, Kitchens and Marcus [10] for a general procedure). This yields the golden shift (no two consecutive zeros) which has entropy $\log \frac{1+\sqrt{5}}{2}$. Therefore by Proposition 1.2:

$$\dim_M(K_T(A)) = \frac{\log 2}{\log 3} + \left(\frac{1}{\log 2} - \frac{1}{\log 3}\right) \log \frac{1+\sqrt{5}}{2} = 0.8875138\ldots$$

Example 4.3: A similar example is given by the substitution in Figure 6.

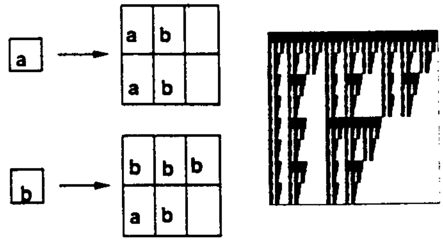

Fig. 6.

One of the matrices is again singular, leading to a calculation like the previous example. The dimension is $\log_2 r = 1.77624...$, where r is the unique positive solution to

$$r = 2^\alpha + \frac{5^\alpha}{r} + \frac{14^\alpha}{r^2} + \cdots + \frac{(\frac{3^k+1}{2})^\alpha}{r^{k-1}} + \cdots.$$

The Minkowski dimension is $1 + \log_2(\frac{3+\sqrt{3}}{2}) = 1.7839...$

Example 4.4: Consider the attractor K for the 3 contractions depicted in Figure 7.

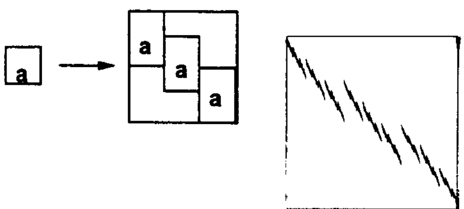

Fig. 7.

To compute $\dim(K)$, observe that $2K$ mod 1 is a sofic set determined by the substitution in Figure 8 and clearly $\dim(2K) = \dim(K)$. One computes the two matrices to be:

$$A_0 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

The Hausdorff dimension is $\dim[K] = 1.36629695\ldots$; the Minkowski dimension is $2 - \log_3(2) = 1.3690702...$

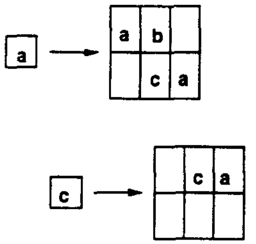

Fig. 8.

What distinguishes this example from those studied in McMullen [25] and Gatzouras and Lalley [17] is the fact that all "Bernoulli" measures on K (obtained by assigning probabilities to the rectangles in the initial pattern) appear to have strictly smaller Hausdorff dimension than K: to compute the Bernoulli measure of maximal dimension, assign probabilities p_1, p_2, p_3 to the three rectangles at positions $(0,1), (1,1/2)$, and $(2,0)$. By symmetry we can assume $p_1 = p_3$, so that $p = (p_1, 1-2p_1, p_1)$. Figure 9 shows two plots of the dimension of μ_p as a function of p_1. In the first, the range of p_1 is from 0 to 0.5, and the dimension of μ_p varies from 0 to its maximum value 1.364987.... In the second plot, p_1 is in the range $[.3, .35]$ and the vertical range is $[1.358, 1.3663]$. The top edge of the box is the value of the Hausdorff dimension of the set in question, strictly larger than largest Bernoulli measure dimension.

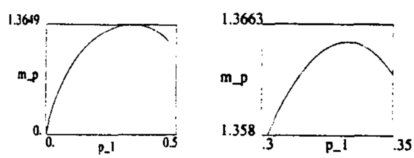

Fig. 9.

Example 4.5: This is a self-affine graph in the sense of Kamae [18] obtained from the following substitution (Figure 10).

Because there is no common eigenvector, the formula in Bedford [4, 5] does not apply. Using Theorem 1.1 the Hausdorff dimension is 1.35912.... The Minkowski dimension is again $2 - \log_3 2$.

Example 4.6: A surprising case of coincidence of Hausdorff and Minkowski dimensions is in Figure 11.

The dimension is $1 + \log_3 2$, the same as the dimension of the product of the ternary Cantor set with an interval; this is easily seen since the above set is just $2K \bmod 1$ where K is the attractor in Figure 12.

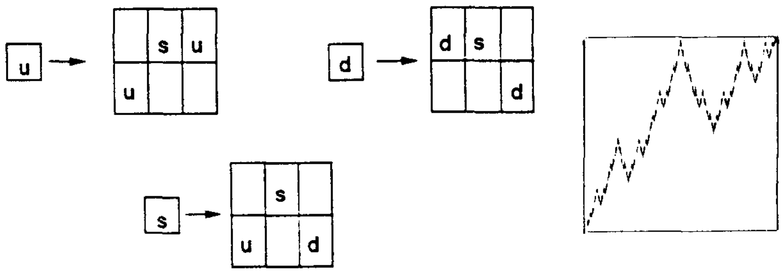

Fig. 10.

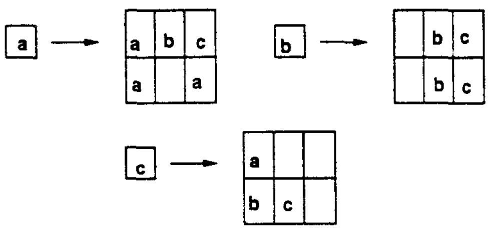

Fig. 11.

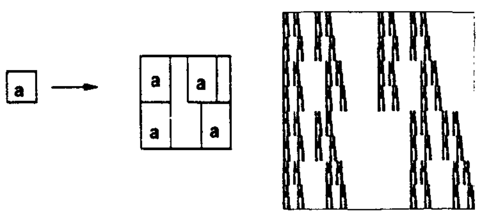

Fig. 12.

Example 4.7: A one-parameter family of self-affine sets. Consider the following one-parameter family of sets.

Let $0 \leq u \leq \frac{1}{2}$ and let K_u be the attractor for the three affine maps

$$L_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x}{3} \\ \frac{y}{2} \end{pmatrix}, \quad L_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x+1}{3} \\ \frac{y}{2}+u \end{pmatrix}, \quad L_3\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x+2}{3} \\ \frac{y+1}{2} \end{pmatrix},$$

that is, K_u is the unique compact set in the plane such that $K_u = \bigcup_{i=1}^{3} L_i(K_u)$. It is easy to see that for all $0 \leq u \leq \frac{1}{2}$,

$$\dim_M(K_u) = 2 - \log_3(2) = 1.36907\ldots$$

(this is included in more general results of Deliu et al. [12] and Falconer [15]).

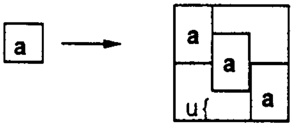

Fig. 13. A one-parameter family of self-affine attractors.

By McMullen [25],

$$\dim(K_0) = \dim(K_{1/2}) = \log_2(1 + 2^{\log_3 2}) = 1.34968\ldots$$

In Example 4.3 we showed that

$$\dim(K_{1/4}) = 1.366297\ldots$$

The same method (but involving more work) may be applied to calculate $\dim(K_u)$ for any rational value of u.

We claim that for almost every $u \in [0, 1/2]$,

$$\dim(K_u) = \dim_M(K_u).$$

Proof: Observe that, after removing a countable subset of K_u, K_u is the graph of a function. Lifting Lebesgue measure from the x-axis to this graph and projecting to the y-axis yields the distribution of the random variable $W = \sum_{n=1}^{\infty} W_n 2^{-n}$ where W_n are i.i.d. random variables taking the values $\{0, 1, u\}$ with equal probability. Consider the random variable

$$Z = \sum_{n=1}^{\infty} Z_n 2^{-n}$$

where the Z_n are i.i.d. and take the values $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with equal probability. Projecting Z to the line connecting the origin to $\begin{pmatrix} 1 \\ u \end{pmatrix}$ yields $\sqrt{1+u^2}W$. The distribution of Z is easily seen to have dimension $\log 3/\log 2 > 1$ (it is essentially Hausdorff measure on the Sierpinski gasket). Thus by Kaufman [19], in almost all directions Z projects to an absolutely continuous random variable, i.e. for a.e. u the set K_u carries a probability measure whose projection to the y-axis (which is the distibution of W) is absolutely continuous. By Bedford and Urbanski [6], theorem 13, it follows that $\dim_M(K_u) = \dim(K_u)$ for such u, proving our claim. ∎

QUESTION: *Is it true that* $\dim_M(K_u) = \dim(K_u)$ *for* **all** *irrational* $u \in [0, 1/2]$?

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